Indian Statistical Institute, Bangalore

B.Math (Hons.) I Year, First Semester

Semestral Examination

Time: 3 hours 2 Dec 2011 Instructor: C.R.E.Raja

Analysis I Maximum marks: 50

Section I: Answer all questions, each question is worth 2 marks

1. If (a_n) and (b_n) are such that $|a_n - b_n| \to 0$ and $a_n \to a$, then prove that $b_n \to a$.
2. Suppose $\sum a_n$ converges and $a_n \geq 0$. Prove that $\sum \sqrt{a_n a_{n+1}}$ also converges.
3. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f'(x) \neq 0$ for any $x \in \mathbb{R}$. Prove that f is one-one.

Section II: Answer any 4 questions, each question is worth 6 marks

1. Let (a_n) be a sequence. Prove that $\limsup a_n$ is a limit point of (a_n).
2. If $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then show that $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ diverges where $s_n = \sum_{k=n}^{\infty} a_k$.
3. Let $f: [a, b] \to \mathbb{R}$ be a continuous function. Suppose f has no local minimum or no local maximum at any point in (a, b). Prove that f is monotonic.
4. Prove that monotonic functions do not have discontinuities of second kind and the set of points of discontinuity is countable.
5. Let $f: \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with $f'(x) = 0$ and $f''(x) < 0$ for some $x \in \mathbb{R}$. Prove that f has a local maximum at x.
6. Let $f: \mathbb{R} \to \mathbb{R}$ be a differentiable function and $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = f(x) - f(x + 1)$ for all $x \in \mathbb{R}$. If $|f'(x)| \to 0$ as $x \to \pm\infty$, prove that g is bounded.

Section III: Answer any 2 questions, each question is worth 10 marks

1. (a) If x is a limit point of (a_n), prove that $\liminf a_n \leq x \leq \limsup a_n$.

   (b) For a sequence (a_n), let $\alpha = \limsup |a_n|^{\frac{1}{n}}$. Prove that $\alpha < 1$ implies $\sum a_n$ converges and $\alpha > 1$ implies $\sum a_n$ does not converge.
2. Let $I = [a, b]$ and $f: I \to \mathbb{R}$ be a continuous function.

   (a) Prove that there are $x, y \in I$ so that $f(x) \leq f(t) \leq f(y)$ for all $t \in I$.

   (b) Further if f is one-one, prove that f is a homeomorphism of I onto $f(I)$.

3. (a) Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions. Prove that for any $r, s \in \mathbb{R}$ there is a t between r and s such that $[f(r) - f(s)]g'(t) = [g(r) - g(s)]f'(t)$.

(b) Let $g(x) = x$ and fix $r \in \mathbb{R}$. Obtain t in (a) as a function of r and s and find $\lim_{s\to r} t$ when $f(x) = x^2$ and $f(x) = x^3$.